

# ONE-COMPONENT INNER FUNCTIONS

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**ABSTRACT.** We explicitly unveil several classes of inner functions  $u$  in  $H^\infty$  with the property that there is  $\eta \in ]0, 1[$  such that the level set  $\Omega_u(\eta) := \{z \in \mathbb{D} : |u(z)| < \eta\}$  is connected. These so-called one-component inner functions play an important role in operator theory.

*Dedicated to the memory of Vadim Tolokonnikov*

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## INTRODUCTION

**Definition 0.1.** *An inner function  $u$  in  $H^\infty$  is said to be a one-component inner function if there is  $\eta \in ]0, 1[$  such that the level set (also called sublevel set or filled level set)  $\Omega_u(\eta) := \{z \in \mathbb{D} : |u(z)| < \eta\}$  is connected.*

One-component inner functions, the collection of which we denote by  $\mathfrak{I}_c$ , were first studied by B. Cohn [10] in connection with embedding theorems and Carleson-measures. It was shown in [10, p. 355] for instance that arclength on  $\{z \in \mathbb{D} : |u(z)| = \varepsilon\}$  is such a measure whenever

$$\Omega_u(\eta) = \{z \in \mathbb{D} : |u(z)| < \eta\}$$

is connected and  $\eta < \varepsilon < 1$ .

A thorough study of the class  $\mathfrak{I}_c$  was given by A.B. Aleksandrov [1] who showed the interesting result that  $u \in \mathfrak{I}_c$  if and only if there is a constant  $C = C(u)$  such that for all  $a \in \mathbb{D}$

$$\sup_{z \in \mathbb{D}} \left| \frac{1 - \overline{u(a)}u(z)}{1 - \bar{a}z} \right| \leq C \frac{1 - |u(a)|^2}{1 - |a|^2}.$$

Many operator-theoretic applications are given in [1, 2, 7, 3]. In our paper here we are interested in explicit examples, which are somewhat lacking in literature. For example, if  $S$  is the atomic inner function, which is given by

$$S(z) = \exp \left( -\frac{1+z}{1-z} \right),$$

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then all level sets  $\Omega_S(\eta)$ ,  $0 < \eta < 1$  are connected, because these sets coincide with the disks

$$(0.1) \quad D_\eta := \left\{ z \in \mathbb{D} : \left| z - \frac{L}{L+1} \right| < \frac{1}{L+1} \right\}, \quad L := \log \frac{1}{\eta},$$

which are tangential to the unit circle at  $p = 1$ .

The scheme of our note here is as follows: in section 1 we prove a general result on level sets which will be the key for our approach to the problem of unveiling classes of one-component inner functions. Then in section 2 we first present with elementary geometric/function theoretic methods several examples and then we use Aleksandrov's criterion to achieve this goal. For instance, we prove that  $BS, B \circ S$  and  $S \circ B$  are in  $\mathfrak{I}_c$  whenever  $B$  is a finite Blaschke product. Considered are also interpolating Blaschke products. It will further be shown that, under the supremum norm,  $\mathfrak{I}_c$  is an open subset of the set of all inner functions and multiplicatively closed. In the final section we give counterexamples.

## 1. LEVEL SETS

We first begin with a topological property of the class of general level sets. Although statement (1) is “well-known” (the earliest appearance seems to be in [26, Theorem VIII, 31]), we could nowhere locate a proof. The argument that the result is a simple and direct consequence of the maximum principle is, in our viewpoint, not tenable.

**Lemma 1.1.** *Given a non-constant inner function  $u$  in  $H^\infty$  and  $\eta \in ]0, 1[$ , let  $\Omega := \Omega_u(\eta) = \{z \in \mathbb{D} : |u(z)| < \eta\}$  be a level set. Suppose that  $\Omega_0$  is a component (=maximal connected subset) of  $\Omega$ . Then*

- (1)  $\Omega_0$  is a simply connected domain; that is,  $\mathbb{C} \setminus \Omega_0$  has no bounded components<sup>1</sup>.
- (2)  $\inf_{\Omega_0} |u| = 0$ .

*Proof.* We show that (1) holds for every holomorphic function  $f$  in  $\mathbb{D}$ ; that is if  $\Omega_0$  is a component of the level set  $\Omega_f(\eta)$ ,  $\eta > 0$ , then it is a simply connected domain<sup>2</sup>. Note that each component  $\Omega_0$  of the open set  $\Omega_f(\eta)$  is an open subset of  $\mathbb{D}$ . We may assume that  $\eta$  is chosen so that  $\{z \in \mathbb{D} : |f(z)| = \eta\} \neq \emptyset$ .

Suppose, to the contrary, that  $D$  is a bounded component of  $\mathbb{C} \setminus \Omega_0$ . Note that  $D$  is closed in  $\mathbb{C}$ . Then, necessarily,  $D$  is contained in  $\mathbb{D}$ , because the unique unbounded complementary component of  $\Omega_0$  contains  $\{z \in \mathbb{C} : |z| \geq 1\}$ . Hence  $D$  is a compact subset of  $\mathbb{D}$ . Let  $G := \Omega_0^*$  be the simply-connected hull of  $\Omega_0$ ; that

<sup>1</sup>A shorter proof can be given by using the advanced definition that a domain  $G$  in  $\mathbb{C}$  is simply connected if every curve in  $G$  is contractible in  $G$ , or equivalently, if for every Jordan curve  $J$  in  $G$  the interior of  $J$  belongs to  $G$ . That depends though on the Jordan curve theorem.

<sup>2</sup>This proof, as well as two different ones, including the one mentioned in footnote 1, stem from the forthcoming book manuscript [22] of the second author together with R. Rupp.

is the union of  $\Omega_0$  with all bounded complementary components of  $\Omega_0$ . Note that  $G$  is open because it coincides with the complement of the unique unbounded complementary component of  $\Omega_0$ . Then, by definition of the simply connected hull,  $D \subseteq G$ . Now if  $H$  is any bounded complementary component of  $\Omega_0$  then (as it was the case for  $D$ )  $H$  is a compact subset of  $\mathbb{D}$  and so  $\partial H \subseteq \mathbb{D}$ . Moreover,

$$(1.1) \quad \partial H \subseteq \partial \Omega_0.$$

In fact, given  $z_0 \in \partial H$ , let  $U$  be a disk centered at  $z_0$ . Then  $U \cap \Omega_0 \neq \emptyset$ , since otherwise  $U \cup H$  would be a connected set strictly bigger than  $H$  and contained in the complement of  $\Omega_0$ ; a contradiction to the maximality of  $H$ . Since  $z_0 \in \partial H \subseteq H \subseteq \mathbb{C} \setminus \Omega_0$ , we conclude that  $z_0 \in \partial \Omega_0$ .

Now  $\partial H \subseteq \partial \Omega_0$  and  $\Omega_0 \subseteq \Omega_f(\eta)$  imply that  $|f| \leq \eta$  on  $\partial H$ , and so, by the maximum principle,  $|f| \leq \eta$  on  $H$ . Consequently,  $|f| \leq \eta$  on  $G$ . By the local maximum principle,  $|f| < \eta$  on  $G$ . Since  $\partial D \subseteq D \subseteq G$ ,

$$(1.2) \quad |f| < \eta \text{ on } \partial D.$$

On the other hand,

$$(1.3) \quad \partial D \stackrel{(1.1)}{\subseteq} \partial \Omega_0 \cap \mathbb{D} \subseteq \{z \in \mathbb{D} : |f(z)| = \eta\}.$$

Note that the second inclusion follows from the fact that if  $|f(z_0)| < \eta$  for  $z_0 \in \partial \Omega_0 \cap \mathbb{D}$ , then  $\Omega_0$  would no longer be a maximal connected subset of  $\Omega_f(\eta)$ . Hence  $|f| = \eta$  on  $\partial D$ . This is a contradiction to (1.2). Thus  $\Omega_0$  is a simply connected domain.

(2) If  $\overline{\Omega}_0 \subseteq \mathbb{D}$ , then, due to  $\partial \Omega_0 \subseteq \{z \in \mathbb{D} : |u(z)| = \eta\}$ , we obtain from the minimum principle that  $u$  must have a zero in  $\Omega_0$ . Now let  $E := \overline{\Omega}_0 \cap \partial \mathbb{D} \neq \emptyset$ . In view of achieving a contradiction, suppose that  $u$  is bounded away from zero in  $\Omega_0$ . Then  $1/|u|$  is subharmonic and bounded in  $\Omega_0$  and

$$\limsup_{\substack{\xi \rightarrow x \\ x \in \partial \Omega_0 \setminus E}} |u(\xi)|^{-1} = \eta^{-1}.$$

Since  $u$  is an inner function,  $E$  has linear measure zero (by [5, Theorem 4.2]). The maximum principle for subharmonic functions with few exceptional points (here on the set  $E$ ; see [6] or [12]), now implies that  $|u|^{-1} \leq \eta^{-1}$  on  $\Omega_0$ . But  $|u| < \eta$  on  $\Omega$  is a contradiction. We conclude that  $\inf_{\Omega_0} |u| = 0$ .  $\square$

**Lemma 1.2.** [10] *Let  $u$  be an inner function. Then the connectedness of  $\Omega_u(\eta)$  implies the one of  $\Omega_u(\eta')$  for every  $\eta' > \eta$ .*

*Proof.* Because  $\Omega_u(\eta)$  is connected and  $\Omega_u(\eta) \subseteq \Omega_u(\eta')$ ,  $\Omega_u(\eta)$  is contained in a unique component  $U_1(\eta')$  of  $\Omega_u(\eta')$ . Suppose that  $U_0(\eta')$  is a second component of  $\Omega_u(\eta')$ . Then  $|u| \geq \eta$  on  $U_0(\eta')$ , because  $U_0(\eta')$  is disjoint with  $U_1(\eta')$  and

hence with  $\Omega_u(\eta)$ . By Lemma 1.1 though,  $\inf_{U_0(\eta')} |u| = 0$ ; a contradiction. Thus  $\Omega_u(\eta')$  is connected.  $\square$

## 2. EXPLICIT EXAMPLES OF ONE-COMPONENT INNER FUNCTIONS

Let

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$$

be the pseudohyperbolic distance of  $z$  to  $w$  in  $\mathbb{D}$  and

$$D_\rho(z_0, r) = \{z \in \mathbb{D} : \rho(z, z_0) < r\}$$

the associated disks,  $0 < r < 1$ . Here is a first class of examples of functions in  $\mathfrak{I}_c$ . Although the next Proposition must be known (in view of A.B. Aleksandrov's criterion [1]), see 2.12 below), we include a simple geometric proof for the reader's convenience.

**Proposition 2.1.** *Let  $B$  be a finite Blaschke product. Then  $B \in \mathfrak{I}_c$ .*

*Proof.* Denote by  $z_1, \dots, z_N$  the zeros of  $B$ , multiplicities included. Let  $\eta \in ]0, 1[$  be chosen so close to 1 that  $G := \bigcup_{n=1}^N D_\rho(z_n, \eta)$  is connected (for example by choosing  $\eta$  so that  $z_j \in D_\rho(z_1, \eta)$  for all  $j$ ). Now

$$G \subseteq \{z \in \mathbb{D} : |B(z)| < \eta\} = \Omega_B(\eta),$$

because  $z \in G$  implies that for some  $n$ ,

$$|B(z)| = \rho(B(z), B(z_n)) \leq \rho(z, z_n) < \sigma.$$

Since  $G$  is connected, there is a unique component  $\Omega_1$  of  $\Omega$  containing  $G$ . In particular,  $Z(B) \subseteq G \subseteq \Omega_1$ . If, in view of achieving a contradiction, we suppose that  $\Omega := \Omega_B(\eta)$  is not connected, there is a component  $\Omega_0$  of  $\Omega$  which is disjoint with  $\Omega_1$ , and so with  $G$ . In particular,

$$(2.1) \quad \rho(z, Z(B)) \geq \sigma \text{ for every } z \in \Omega_0.$$

Since  $\overline{\Omega_0} \subseteq \overline{\Omega_B(\eta)} \subseteq \mathbb{D}$ , and  $|B| = \eta$  on  $\partial\Omega_0$ , we deduce from the minimum principle that  $\Omega_0$  contains a zero of  $B$ ; a contradiction.  $\square$

We now generalize this result to a class of interpolating Blaschke products. Recall that a Blaschke product  $b$  with zero set/sequence  $\{z_n : n \in \mathbb{N}\}$  is said to be an interpolating Blaschke product if  $\delta(b) := \inf(1 - |z_n|^2)|b'(z_n)| > 0$ . If  $b$  is an interpolating Blaschke product then, for small  $\varepsilon$ , the pseudohyperbolic disks

$$D_\rho(z_n, r) = \{z \in \mathbb{D} : \rho(z, z_n) < \varepsilon\}$$

are pairwise disjoint. Moreover, by Hoffman's Lemma (see below and also [19]), for any  $\eta \in ]0, 1[$ ,  $b$  is bounded away from zero on  $\{z \in \mathbb{D} : \rho(z, Z(b)) \geq \eta\}$ .

**Theorem 2.2** (Hoffman's Lemma). ([18] p. 86, 106 and [13] p. 404, 310). Let  $\delta, \eta$  and  $\epsilon$  be real numbers, called Hoffman constants, satisfying  $0 < \delta < 1$ ,  $0 < \eta < (1 - \sqrt{1 - \delta^2})/\delta$ , (that is,  $0 < \eta < \rho(\delta, \eta)$ ) and

$$0 < \epsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

If  $B$  is any interpolating Blaschke product with zeros  $\{z_n : n \in \mathbb{N}\}$  such that

$$\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2) |B'(z_n)| \geq \delta,$$

then

- 1) the pseudohyperbolic disks  $D_\rho(a, \eta)$  for  $a \in Z(B)$  are pairwise disjoint.
- (2) The following inclusions hold:

$$\{z \in \mathbb{D} : |B(z)| < \epsilon\} \subseteq \{z \in \mathbb{D} : \rho(z, Z(B)) < \eta\} \subseteq \{z \in \mathbb{D} : |B(z)| < \eta\}.$$

A large class of interpolating Blaschke products which are one-component inner functions now is given in the following result.

**Theorem 2.3.** Let  $b$  be an interpolating Blaschke product with zero set  $\{z_n : n \in \mathbb{N}\}$ . Suppose that for some  $\sigma \in ]0, 1[$  the set

$$G := \bigcup_n D_\rho(z_n, \sigma)$$

is connected. Then  $b$  is a one-component inner function. This holds in particular, if  $\rho(z_n, z_{n+1}) < \sigma < 1$  for all  $n$ ; for example if  $z_n = 1 - 2^{-n}$ .

*Proof.* As in the proof of Proposition 2.1

$$G \subseteq \{z \in \mathbb{D} : |b(z)| < \sigma\} =: \Omega.$$

Since  $G$  is assumed to be connected, there is a unique component  $\Omega_1$  of  $\Omega$  containing  $G$ . In particular,  $Z(b) \subseteq G \subseteq \Omega_1$ . Now, if we suppose that  $\Omega$  is not connected, then there is a component  $\Omega_0$  of  $\Omega$  which is disjoint with  $\Omega_1$ , and so with  $G$ . In particular,

$$(2.2) \quad \rho(z, Z(b)) \geq \sigma \text{ for every } z \in \Omega_0.$$

Let  $\delta := \delta(b)$ ,

$$0 < \eta < \min\{(1 - \sqrt{1 - \delta^2})/\delta, \sigma\},$$

$$0 < \epsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

By Lemma 1.1,  $\inf_{\Omega_0} |b| = 0$ . Thus, there is  $z_0 \in \Omega_0$  be so that  $|b(z_0)| < \epsilon$ . We deduce from Hoffman's Lemma 2.2 that  $\rho(z_0, Z(b)) < \eta < \sigma$ . This is a contradiction to (2.2). We conclude that  $\Omega$  must be connected. It is clear that

the condition  $\rho(z_n, z_{n+1}) < \sigma$  for every  $n$  implies that  $\bigcup_n D_\rho(z_n, \sigma)$  is connected. For the rest, just note that

$$\rho(1 - 2^{-n}, 1 - 2^{-n-1}) = \frac{2^{-n} - 2^{-n-1}}{2^{-n} + 2^{-n-1} + 2^{-n}2^{-n-1}} = \frac{1}{3 + 2^{-n}}.$$

□

**Corollary 2.4.** *Let  $B$  be a Blaschke product with increasing real zeros  $x_n$  satisfying*

$$0 < \eta_1 \leq \rho(x_n, x_{n+1}) \leq \eta_2 < 1.$$

*Then  $b \in \mathfrak{I}_c$ .*

*Proof.* Just use Theorem 2.3 and the fact that by the Vinogradov-Hayman-Newman theorem,  $B$  is interpolating if and only if

$$\sup_n \frac{1 - x_{n+1}}{1 - x_n} \leq s < 1$$

or equivalently

$$\inf_n \rho(x_n, x_{n+1}) \geq r > 0.$$

□

Using a result of Kam-Fook Tse [25], telling us that a sequence  $(z_n)$  of points contained in a Stolz angle (or cone)  $\{z \in \mathbb{D} : |1 - z| < C(1 - |z|)\}$  is interpolating if and only if it is separated (meaning that  $\inf_{n \neq m} \rho(z_n, z_m) > 0$ ), we obtain:

**Corollary 2.5.** *Let  $B$  be a Blaschke product whose zeros  $(z_n)$  are contained in a Stolz angle and are separated. Suppose that  $\rho(z_n, z_{n+1}) \leq \eta < 1$ . Then  $B \in \mathfrak{I}_c$ .*

Similarly, using a result by M. Weiss [27, Theorem 3.6] and its refinement in [4, Theorem B], we also obtain the following assertion for sequences that may be tangential at 1 (see also Wortman [28]).

**Corollary 2.6.** *Let  $B$  be a Blaschke product whose zeros  $z_n = r_n e^{i\theta_n}$  satisfy:*

$$\begin{aligned} r_n &< r_{n+1}, \quad \theta_{n+1} < \theta_n, \\ r_n &\nearrow 1, \quad \theta_n \searrow 0, \end{aligned}$$

$$(2.3) \quad 0 < \eta_1 \leq \rho(z_n, z_{n+1}) \leq \eta_2 < 1.$$

*Then  $B$  is an interpolating Blaschke product contained in  $\mathfrak{I}_c$ . This holds in particular if the zeros are located on a convex curve in  $\mathbb{D}$  with endpoint 1 and satisfying (2.3).*

Other classes of this type can be deduced from [14]. Here are two explicit examples of interpolating Blaschke products in  $\mathfrak{I}_c$  whose zeros are given by iteration of the zero of a hyperbolic, respectively parabolic automorphism of  $\mathbb{D}$ . These functions appear, for instance, in the context of isometries on the Hardy space  $H^p$  (see [8]).

**Example 2.7.** • Let  $\varphi(z) = \frac{z - 1/2}{1 - (1/2)z}$ . Then  $\varphi$  is an hyperbolic automorphism with fixed points  $\pm 1$ . If  $\varphi_j := \underbrace{\varphi \circ \cdots \circ \varphi}_{j\text{-times}}$ , then  $\varphi_j \in \text{Aut}(\mathbb{D})$  and vanishes exactly at the point

$$x_j := \frac{3^j - 1}{3^j + 1} = 1 - \frac{2}{3^j + 1}.$$

This can readily be seen by using that  $x_{j+1} = \varphi^{-1}(x_j)$  and

$$\varphi_{j+1}(z) = (\varphi_j \circ \varphi)(z) = \frac{z - \frac{\frac{1}{2} + x_j}{1 + \frac{1}{2}x_j}}{1 - z \frac{\frac{1}{2} + x_j}{1 + \frac{1}{2}x_j}}.$$

Since

$$\rho(x_j, x_{j+1}) = \frac{3^{j+1} - 3^j}{3^{j+1} + 3^j} = \frac{1}{2},$$

we deduce from Corollary 2.4 that the Blaschke product

$$B(z) := \prod_{j=1}^{\infty} \frac{x_j - z}{1 - x_j z}$$

associated with these zeros is in  $\mathfrak{I}_c$ .

• Let  $\sigma \in \text{Aut}(\mathbb{D})$  and  $\tau = \sigma \circ \varphi \circ \sigma^{-1}$ . Then  $\tau$  is also a hyperbolic automorphism fixing the points  $\sigma(\pm 1)$ , and where  $\xi := \sigma(1)$  is the Denjoy-Wolff point with  $\tau'(\xi) < 1$ . The zeros of the  $n$ -th iterate  $\tau_n$  of  $\tau$  are given by

$$z_n = \tau_n^{-1}(0) = (\sigma \circ \varphi_n^{-1} \circ \sigma^{-1})(0).$$

By the grand iteration theorem [23, p.78], since 1 is an attracting fixpoint with  $(\varphi^{-1})'(1) = 1/3 < 1$ , the sequence  $(\varphi_n^{-1}(\sigma^{-1}(0)))$  converges nontangentially to 1. Hence the points  $z_n$  are located in a cone with cusp at  $\xi$ . Moreover, if  $n > k$  and  $a = \sigma^{-1}(0)$ ,

$$\begin{aligned} \rho(z_n, z_k) &= \rho((\varphi_n^{-1} \circ \sigma^{-1})(0), (\varphi_k^{-1} \circ \sigma^{-1})(0)) \\ &= \rho(\varphi_{n-k}^{-1}(a), a) \end{aligned}$$

Thus,  $\rho(z_n, z_{n+1}) = \rho(\varphi(a), a)$  for all  $n$  and  $\inf_{n \neq k} \rho(z_n, z_k) > 0$ . Now  $(z_n)$  is a Blaschke sequence<sup>3</sup> ([23, Ex. 6, p. 85]); in fact, use d'Alembert's quotient criterion and observe that by the Denjoy-Wolff theorem,

$$\frac{1 - |z_{n+1}|}{1 - |z_n|} = \frac{1 - |\tau^{-1}(z_n)|}{1 - |z_n|} \rightarrow (\tau^{-1})'(\xi) < 1.$$

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<sup>3</sup> This also follows from the inequalities  $1 - |\sigma(\xi_n)|^2 = \frac{(1 - |a|^2)(1 - |\xi_n|^2)}{|1 - \bar{a}\xi_n|^2} \leq \frac{1 + |a|}{1 - |a|}(1 - |\xi_n|^2)$  and  $1 - |\psi_n(a)|^2 \leq \frac{1 + |a|}{1 - |a|}(1 - |w_n|^2)$ , whenever  $(w_n)$  is a Blaschke sequence and  $\psi_n(w_n) = \sigma(a) = 0$ .

Hence, by Corollary 2.5,  $(z_n)$  is an interpolating sequence (see also [11, p.80]) and the associated Blaschke product  $b = \prod_{n=1}^{\infty} e^{i\theta_n} \tau_n$  belongs to  $\mathfrak{I}_c$  (here  $\theta_n$  is chosen so that the  $n$ -th Blaschke factor is positive at the origin).

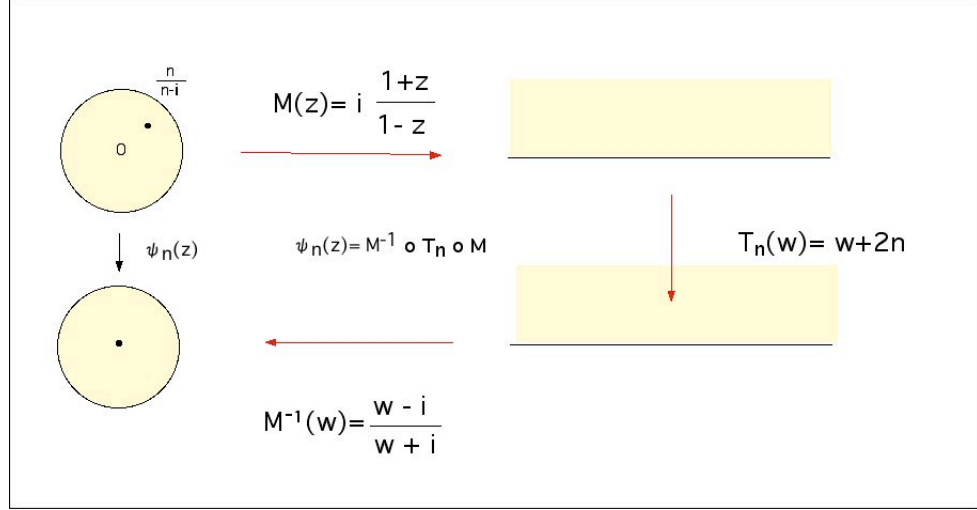


FIGURE 1. The parabolic automorphism

• Let  $\psi(z) = i \frac{z - \frac{1+i}{2}}{1 - \frac{1-i}{2}z}$ . Then  $\psi$  is a parabolic automorphism with attracting fixed point 1. The automorphism  $\psi$  is conjugated to the translation  $w \mapsto w + 2$  on the upper half-plane (see figure 1) via the map  $M(z) = i(1+z)/(1-z)$  and  $\psi_n = M^{-1} \circ T_n \circ M$ . The zeros of the  $n$ -th iterate  $\psi_n$  of  $\psi$  are given by

$$z_n = \frac{n}{n-i};$$

just use that  $z_n = (M^{-1} \circ T_n^{-1} \circ M)(0)$ . These zeros satisfy  $|z_n - \frac{1}{2}| = \frac{1}{2}$  and of course also the Blaschke condition  $\sum_{n=1}^{\infty} 1 - |z_n|^2 < \infty$ . Moreover,

$$\rho(z_n, z_{n+1}) = \frac{1}{\sqrt{2}}.$$

Thus, by, Corollary 2.6, the Blaschke product associated with these zeros is interpolating and belongs to  $\mathfrak{I}_c$ .

**Proposition 2.8.** *Let  $B$  be a finite Blaschke product or an interpolating Blaschke product with real zeros clustering at  $p = 1$ . Then  $f := BS \in \mathfrak{I}_c$ .*

*Proof.* i) Let  $B$  be a finite Blaschke product. Chose  $\eta \in ]0, 1[$  so close to 1 that the disk  $D_\eta$  in (0.1), which coincides with the level set  $\Omega_S(\eta)$ , contains all zeros of  $B$ . Now  $D_\eta = \Omega_S(\eta) \subseteq \Omega_f(\eta)$ . Now  $\Omega_f(\eta)$  must be connected, since otherwise there



would be a component  $\Omega_0$  of  $\Omega_f(\eta)$  disjoint from the component  $\Omega_1$  containing  $D_\eta$ . But  $f$  is bounded away from zero outside  $D_\eta$ ; hence  $f = BS$  is bounded away from zero on  $\Omega_0$ . This is a contradiction to Lemma 1.1 (2).

ii) If  $B$  is an interpolating Blaschke product with zeros  $(z_n)$ , then, by Hoffman's Lemma 2.2,  $B$  is bounded away from zero outside  $R := \bigcup D_\rho(z_n, \varepsilon)$  for every  $\varepsilon \in ]0, 1[$ . Now, if the zeros of  $B$  are real, and bigger than  $-\sigma$  for some  $\sigma \in ]0, 1[$ , this set  $R$  is contained in a cone with cusp at 1 and aperture-angle strictly less than  $\pi$  (see for instance [21]). Hence  $R$  is contained in  $D_\eta$  for all  $\eta$  close to 1. Thus, as above, we can deduce that  $\Omega_{BS}(\eta)$  is connected.  $\square$

The previous result shows, in particular, that certain non one-component inner functions (for example a thin Blaschke product with positive zeros, see Corollary 3.1), can be multiplied by a one-component inner function into  $\mathfrak{I}_c$ . In particular,  $uv \in \mathfrak{I}_c$  does not imply that  $u$  and  $v$  belong to  $\mathfrak{I}_c$ . The reciprocal, though, is true: that is  $\mathfrak{I}_c$  itself is stable under multiplication, as we are going to show below.

**Proposition 2.9.** *Let  $u, v$  be two inner functions in  $\mathfrak{I}_c$ . Then  $uv \in \mathfrak{I}_c$ .*

*Proof.* Let  $\Omega_u(\eta)$  and  $\Omega_v(\eta')$  be two connected level sets. Due to monotonicity (Lemma 1.2), and the fact that  $\bigcup_{\lambda \in [\lambda_0, 1[} \Omega_f(\lambda) = \mathbb{D}$ , we may assume that  $\sigma$  satisfies

$$\max\{\eta, \eta'\} \leq \sigma < 1$$

and is so close to 1 that  $0 \in \Omega_u(\sigma) \cap \Omega_v(\sigma) \neq \emptyset$ . Hence  $U := \Omega_u(\sigma) \cup \Omega_v(\sigma)$  is connected. Now

$$\Omega_u(\sigma) \cup \Omega_v(\sigma) \subseteq \Omega_{uv}(\sigma).$$

If we suppose that  $\Omega_{uv}(\sigma)$  is not connected, then there is a component  $\Omega_0$  of  $\Omega_{uv}(\sigma)$  which is disjoint from  $U$ . In particular,  $u$  and  $v$  are bounded away from zero on  $\Omega_0$ . This contradicts Lemma 1.1 (2). Hence  $\Omega_{uv}(\sigma)$  is connected and so  $uv \in \mathfrak{I}_c$ .  $\square$

**Theorem 2.10.** *The set of one-component inner functions is open inside the set of all inner functions (with respect to the uniform norm topology).*

*Proof.* Let  $u \in \mathfrak{I}_c$ . Then, by Lemma 1.2,  $\Omega_u(\eta)$  is connected for all  $\eta \in [\eta_0, 1[$ . Choose  $0 < \varepsilon < \min\{\eta, 1 - \eta\}$  and let  $\Theta$  be an inner function with  $\|u - \Theta\| < \varepsilon$ . We claim that  $\Theta \in \mathfrak{I}_c$ , too. To this end we note that

$$\Omega_\Theta(\eta - \varepsilon) \subseteq \Omega_u(\eta) \subseteq \Omega_\Theta(\eta + \varepsilon),$$

where  $0 < \eta - \varepsilon < \eta + \varepsilon < 1$ . As usual, if we suppose that  $\Omega_\Theta(\eta + \varepsilon)$  is not connected, then there is a component  $\Omega_0$  of  $\Omega_\Theta(\eta + \varepsilon)$  which is disjoint from the connected set  $\Omega_u(\eta)$ , hence disjoint with  $\Omega_\Theta(\eta - \varepsilon)$ . In other words,  $|\Theta| \geq \eta - \varepsilon > 0$  on  $\Omega_0$ . This contradicts Lemma 1.1 (2). Hence  $\Omega_\Theta(\eta + \varepsilon)$  is connected and so  $\Theta \in \mathfrak{I}_c$ .  $\square$

Next we look at right-compositions of  $S$  with finite Blaschke products. We first deal with the case where  $B(z) = z^2$ .

**Example 2.11.** *The function  $S(z^2)$  is a one-component inner function.*

*Proof.* Let  $\Omega_S(\eta)$  be the  $\eta$ -level set of  $S$ . Then, as we have already seen, this is a disk tangent to the unit circle at the point 1. We may choose  $0 < \eta < 1$  so close to 1 that 0 belongs to  $\Omega_S(\eta)$ . Let  $U = \Omega_S(\eta) \setminus ]-\infty, 0]$ . Then  $U$  is a simply connected slitted disk on which exists a holomorphic square root  $q$  of  $z$ . The image of  $U$  under  $q$  is a simply connected domain  $V$  in the semi-disk  $\{z : |z| < 1, \operatorname{Re} z > 0\}$ . Let  $V^*$  be its reflection along the imaginary axis. Then  $E := \overline{V^*} \cup \overline{V}$  is mapped by  $z^2$  onto the closed disk  $\overline{\Omega_S(\eta)}$ . Then  $E \setminus \partial E$  coincides with  $\Omega_{S(z^2)}(\eta)$ .

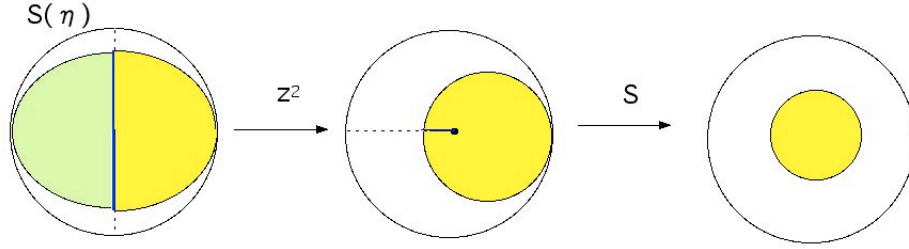


FIGURE 2. The level sets of  $S(z^2)$

□

Using Aleksandrov's criterion (see below), we can extend this by replacing  $z^2$  with any finite Blaschke product. Recall that the spectrum  $\rho(\Theta)$  of an inner function  $\Theta$  is the set of all boundary points  $\zeta$  for which  $\Theta$  does not admit a holomorphic extension; or equivalently, for which  $Cl(\Theta, \zeta) = \overline{\mathbb{D}}$ , where

$$Cl(\Theta, \zeta) = \{w \in \mathbb{C} : \exists (z_n) \in \mathbb{D}^{\mathbb{N}}, \lim z_n = \zeta \text{ and } \lim \Theta(z_n) = w\}$$

is the cluster set of  $\Theta$  at  $\zeta$  (see [13, p. 80]).

**Theorem 2.12** (Aleksandrov). [1, Theorem 1.11 and Remark 2, p. 2915] *Let  $\Theta$  be an inner function. The following assertions are equivalent:*

- (1)  $\Theta \in \mathfrak{I}_c$ .
- (2) *There is a constant  $C > 0$  such that for every  $\zeta \in \mathbb{T} \setminus \rho(\Theta)$  we have*

$$i) |\Theta''(\zeta)| \leq C |\Theta'(\zeta)|^2,$$

and

$$ii) \liminf_{r \rightarrow 1} |\Theta(r\zeta)| < 1 \text{ for all } \zeta \in \rho(\Theta).$$

Note that, due to this theorem,  $\Theta \in \mathfrak{I}_c$  necessarily implies that  $\rho(\Theta)$  has measure zero.

**Proposition 2.13.** *Let  $B$  be a finite Blaschke product. Then  $S \circ B \in \mathfrak{I}_c$ .*

*Proof.* Let us note first that  $\rho(S \circ B) = B^{-1}(\{1\})$ . Since the derivative of  $B$  on the boundary never vanishes (due to

$$(2.4) \quad z \frac{B'(z)}{B(z)} = \sum_{n=1}^N \frac{1 - |a_n|^2}{|a_n - z|^2}, \quad |z| = 1, B(a_n) = 0,)$$

$B$  is schlicht in a neighborhood of 1. The angle conservation law now implies that for  $\zeta \in B^{-1}(1)$  the curve  $r \mapsto B(r\zeta)$  stays in a Stolz angle at 1 in the image space of  $B$ . Hence  $\liminf_{r \rightarrow 1} S(B(r\zeta)) = 0$  for  $\zeta \in \rho(S \circ B)$ . Now let us calculate the derivatives:

$$\begin{aligned} S'(z) &= -S(z) \frac{2}{(1-z)^2}, \\ S''(z) &= S(z) \left[ \frac{4}{(1-z)^4} - \frac{4}{(1-z)^3} \right], \\ (S \circ B)' &= (S' \circ B)B' \\ (S \circ B)'' &= (S'' \circ B)B'^2 + (S' \circ B)B'' \end{aligned}$$

$$(2.5) \quad \begin{aligned} A := \frac{(S \circ B)''}{[(S \circ B)']^2} &= \frac{S'' \circ B}{(S' \circ B)^2} + \frac{(S' \circ B)}{(S' \circ B)^2} \frac{B''}{B'^2} \\ &= \frac{S'' \circ B}{(S' \circ B)^2} + \frac{1}{S' \circ B} \frac{B''}{B'^2}. \end{aligned}$$

Hence, for  $\zeta \in \mathbb{T} \setminus \rho(S \circ B)$ ,  $|B(\zeta)| = 1$ , but  $\xi := B(\zeta) \neq 1$ , and so, by (2.4),

$$\begin{aligned} |A(\zeta)| &\leq \sup_{\xi \neq 1} \frac{|S''(\xi)|}{|S'(\xi)|^2} + 2 \sup_{\xi \neq 1} \frac{|1 - \xi|^2}{|S(\xi)|} C \\ &\leq C' \sup_{\xi \neq 1} \frac{|1 - \xi|^4}{|1 - \xi|^4} + 8C < \infty. \end{aligned}$$

□

**Corollary 2.14.** *Let  $S_\mu$  be a singular inner function with finite spectrum  $\rho(S_\mu)$ . Then  $S_\mu \in \mathfrak{I}_c$ .*

*Proof.* Since  $S$  is the universal covering map of  $\mathbb{D} \setminus \{0\}$ , each singular inner function  $S_\mu$  writes as  $S_\mu = S \circ v$  for some inner function  $v$ . Since  $\rho(S_\mu)$  is finite,  $v$  necessarily is a finite Blaschke product. (This can also be seen from [15, Proof of Theorem 2.2]). The assertion now follows from Proposition 2.13. □

Note that this result also follows in an elementary way from Proposition 2.9 and the fact that every such  $S_\mu$  is a finite product of powers of the atomic inner function  $S$ . We now consider left-compositions with finite Blaschke products.

**Proposition 2.15.** *Let  $\Theta$  be a one-component inner function. Then each Frostman shift  $(a - \Theta)/(1 - \bar{a}\Theta) \in \mathfrak{I}_c$ , too. Here  $a \in \mathbb{D}$ .*

*Proof.* Let  $\tau(z) = (a - z)/(1 - \bar{a}z)$ . Then  $\rho(\tau \circ \Theta) = \rho(\Theta)$ . As above,

$$\liminf_{r \rightarrow 1} |\tau \circ \Theta(r\zeta)| < 1$$

for every  $\zeta \in \rho(\tau \circ \Theta)$ . Now

$$\tau(z) = \frac{1}{\bar{a}} + \frac{|a|^2 - 1}{\bar{a}} \frac{1}{1 - \bar{a}z},$$

from which we easily deduce the first and second derivatives. By using the formulas 2.5, we obtain

$$A := \left| \frac{(\tau \circ \Theta)''}{[(\tau \circ \Theta)']^2} \right| \leq C \frac{|1 - \bar{a}\Theta|^4}{|1 - \bar{a}\Theta|^3} + C'|1 - \bar{a}\Theta|^2 \frac{|\Theta''|}{|\Theta'|^2}.$$

Hence, the assumption  $\Theta \in \mathfrak{I}_c$  now yields (via Aleksandrov's criterion 2.12) that  $\sup_{\zeta \in \rho(\tau \circ \Theta)} A(\zeta) < \infty$ . Thus  $\tau \circ \Theta \in \mathfrak{I}_c$ . □

**Corollary 2.16.** *Given  $a \in \mathbb{D} \setminus \{0\}$ , the interpolating Blaschke products  $(S - a)/(1 - \bar{a}S)$  belong to  $\mathfrak{I}_c$ .*

This also follows from Corollary 2.6 by noticing that the  $a$ -points of  $S$  are located on a disk tangent at 1 and that the pseudohyperbolic distance between two consecutive ones is constant (see [20]). There it is also shown that the Frostman shift  $(S - a)/(1 - \bar{a}S)$  is an interpolating Blaschke product.

**Corollary 2.17.** *Let  $B$  be a finite Blaschke product and  $\Theta \in \mathfrak{I}_c$ . Then  $B \circ \Theta \in \mathfrak{I}_c$ .*

*Proof.* This is a combination of Propositions 2.15 and 2.9. □

### 3. INNER FUNCTIONS NOT BELONGING TO $\mathfrak{I}_c$

Here we present a class of Blaschke products that are not one-component inner functions. Recall that a Blaschke product  $b$  with zero-sequence  $(z_n)$  is *thin* if

$$\lim_n \prod_{k \neq n} \rho(z_k, z_n) = \lim_{n \rightarrow 1} (1 - |z_n|^2) |b'(z_n)| = 1.$$

It was shown by Tolokonnikov [24, Theorem 2.3] that  $b$  is thin if and only if

$$\lim_{|z| \rightarrow 1} (|b(z)|^2 + (1 - |z|^2) |b'(z)|) = 1.$$

**Corollary 3.1.** *Thin Blaschke products are never one-component inner functions.*

*Proof.* Let  $\varepsilon \in ]0, 1[$  be arbitrary close to 1. Choose  $\eta > 0$  and  $\delta > 0$  so close to 1 so that

$$\varepsilon < \eta^2 \text{ and } \eta < (1 - \sqrt{1 - \delta^2})/\delta.$$

By deleting finitely many zeros, say  $z_1, \dots, z_N$  of  $b$ , we obtain a tail  $b_N$  such that  $(1 - |z_n|^2)|b'_N(z_n)| \geq \delta$  for every  $n > N$ . Hence, by Theorem 2.2,

$$(3.1) \quad \{z \in \mathbb{D} : |b_N(z)| < \varepsilon\} \subseteq \{z \in \mathbb{D} : \rho(z, Z(b_N)) < \eta\}$$

and the disks  $D(z_n, \eta)$  are pairwise disjoint. This implies that the level set  $\{z \in \mathbb{D} : |b_N(z)| < \varepsilon\}$  is not connected. Now choose  $r$  so close to 1 that

$$p(z) := \prod_{n=1}^N \rho(z, z_n) \geq \varepsilon$$

for every  $z$  with  $r \leq |z| < 1$ . We show that the level set  $\{|b| < \varepsilon^2\}$  is not connected. In fact, for some  $r \leq |z| < 1$  we have  $|b(z)| < \varepsilon^2$ , then

$$|b_N(z)| = \frac{|b(z)|}{|p(z)|} < \frac{\varepsilon^2}{\varepsilon} = \varepsilon.$$

Hence

$$\{z : r < |z| < 1, |b(z)| < \varepsilon^2\} \subseteq \{|b_N(z)| < \varepsilon\} \stackrel{(3.1)}{\subseteq} \bigcup_{n>N} D(z_n, \eta).$$

Since the disks  $D_\rho(z_n, \eta)$  are pairwise disjoint if  $n > N$ , we are done.  $\square$

**Corollary 3.2.** *No finite product  $B$  of thin interpolating Blaschke products belongs to  $\mathfrak{I}_c$ .*

*Proof.* Let  $\varepsilon \in ]0, 1[$  be arbitrary close to 1. By Corollary 3.1, if  $b_j$ , ( $j = 1, 2$ ), are two thin Blaschke products with zero-sequence  $(z_n^{(j)})_n$ ,

$$\Omega_{b_j}(\varepsilon) \subseteq \bigcup_{n=1}^{\infty} D_\rho(z_n^{(j)}, \eta)$$

for suitable  $\eta$ , the disks  $D_\rho(z_n^{(j)}, \eta)$ , being pairwise disjoint for  $n$  large. Since  $\lim_n \rho(z_n^{(j)}, z_{n+1}^{(j)}) = 1$ , we see that a disk  $D_\rho(z_n^{(1)}, \eta)$  can meet at most one disk  $D_\rho(z_m^{(2)}, \eta)$  for  $n$  large. Hence

$$\Omega_{b_1 b_2}(\varepsilon^2) \subseteq \bigcup_{j=1}^2 \bigcup_{n=1}^{\infty} D_\rho(z_n^{(j)}, \eta),$$

where the set on the right hand side obviously is disconnected. The general case works via induction.  $\square$

**Remark.** *The conditions*

$$(3.2) \quad \eta^* := \sup_{n \in \mathbb{N}} \rho(z_n, Z(b) \setminus \{z_n\}) < 1,$$

*or equivalently*

$$(3.3) \quad D(z_n, \eta) \cap \bigcup_{m \neq n} D(z_m, \eta) \neq \emptyset \text{ for some } \eta \in ]0, 1[,$$

*are not sufficient to guarantee that the interpolating Blaschke product  $b$  is a one-component inner function.*

*Proof.* Take  $z_{2n} = 1 - n^{-n}$  and  $z_{2n+1} = 1 - (n^{-n} + n^{-n})$ . Then  $(z_{2n})$  and  $(z_{2n+1})$  are (thin) interpolating sequences by [16, Corollary 2.4]. Using with  $a = n^{-n}$  and  $b = 2a$  the identity

$$\rho(1 - a, 1 - b) = \frac{|a - b|}{a + b - ab},$$

we conclude that

$$\rho(z_{2n}, z_{2n+1}) = \frac{n^{-n}}{1 - z_{2n}z_{2n+1}} \rightarrow 1/3,$$

and so the union  $(z_n)$  is an interpolating sequence satisfying (3.3). By Corollary 3.2, the Blaschke product formed with the zero-sequence  $(z_n)$  is not in  $\mathfrak{I}_c$ .  $\square$

Using the following theorem in [5], we can exclude a much larger class of Blaschke products from being one-component inner functions:

**Theorem 3.3** (Berman). *Let  $u$  be an inner function. Then, for every  $\varepsilon \in ]0, 1[$ , all the components of the level sets  $\{z \in \mathbb{C} : |u(z)| < \varepsilon\}$  have compact closures in  $\mathbb{D}$  if and only if  $u$  is a Blaschke product and*

$$\limsup_{r \rightarrow 1} |u(r\xi)| = 1 \text{ for every } \xi \in \mathbb{T}.$$

In particular this condition is satisfied by finite products of thin Blaschke products (see [17, Proposition 2.2]) as well as by the class of uniform Frostman Blaschke products

$$\sup_{\xi \in \mathbb{T}} \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|\xi - z_n|} < \infty.$$

Note that this Frostman condition implies that the associated Blaschke product has radial limits of modulus one everywhere [9, p. 33]. As a byproduct of Theorem 2.3 we therefore obtain

**Corollary 3.4.** *If  $b$  is a uniform Frostman Blaschke product with zeros  $(z_n)$  clustering at a single point, then  $\limsup_{\rho}(z_n, z_{n+1}) = 1$ .*

**Questions 3.5.** To conclude, we would like to ask two questions and present three problems:

- (1) *Can every inner function  $u$  whose boundary spectrum  $\rho(u)$  has measure zero, be multiplied by a one-component inner function into  $\mathfrak{I}_c$ ?*
- (2) *Let  $S_\mu$  be a singular inner function with countable spectrum. Give a characterization of those measures  $\mu$  such that  $S_\mu \in \mathfrak{I}_c$ . Do the same for singular continuous measures.*
- (3) *In terms of the zeros, give a characterization of those interpolating Blaschke products that belong to  $\mathfrak{I}_c$ .*
- (4) *Does the Blaschke product  $B$  with zeros  $z_n = 1 - n^{-2}$  belong to  $\mathfrak{I}_c$ ?*

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